An Integral Representation of Wiener-Shannon Formula

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An integral representation of Wiener-Shannon formula is derived from the Euler's integral representation of Gaussian hypergeometric function. The formula accounts for the available speed of information transmission through band-limited, additive white Gaussian channels. The representation allows one to define, analogously to Newton mechanics, the instantaneous acceleration of information flow in the channels.

In recent years, concepts of Shannon's information theory (Shannon & Weaver, 1949) were applied to various contexts of physics. Typical examples are seen in quantum-mechanical problems (Partovi, 1990: 357; Bialynicki-Birula et al., 1992: 75), random matrix theory of Gaussian orthogonal ensembles (Ichimura et al., 1993: 80), statistical analyses of strange attractors and fractals (Crutchfield & Packard, 1982; Grassberger & Procaccia, 1984; Wales, 1991), DNA linguistics (Mantegna et al., 1994), nuclear giant resonances (Drożdż et al., 1995), and quantum optics (Vaccaro & Orlowski, 1995). It is worth stressing here that, of many formulas presented by Shannon, one of the most famous and beautiful relations is that termed Wiener-Shannon formula (WSF)(Shannon & Weaver, 1949; Fano, 1961: 159). [Note that in a certain context this formula was termed differently. For instance, Brillouin terms this Hartley-Tuller-Shannon formula, Tuller-

Shannon formula for short (Brillouin, 1956).] The formula accounts for the available speed of information transmission through band-limited, additive white Gaussian channels. It is interesting to note that more recently relations similar to the original WSF were given in the context of dynamical chaos (Kravtsov, 1989: 44) and chaotic information processing (Nara & Davis, 1990: 230) as well as in that of fuzzy set theory (Klir & Yuan, 1995: 60). Since the WSF provides the speed of information transmission, it may be natural for physicists to pose a question: Is there a concept corresponding to the acceleration of information transmission? If so, how is it definable? In this Short Article, through derivation of a compact integral representation of the WSF, it is shown that an effective acceleration of information flow in the channels can be introduced in information theory.

According to Fano's notation (Fano, 1961: 159) the WSF can be presented in the

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following theorem: If the input probability distribution is subject to the requirement that the ensemble average of the time average of $u^2(t)$ be at most equal to some value S, that is,

$$\int_U | \underline{u} |^2 \, p(\underline{u}) \, dU \leq S, \tag{1}$$

the capacity of a band-limited channel with additive white Gaussian noise is

$$C = W \log_2 [1 + S/(N_0 W)] \text{ bit/sec}, \tag{2}$$

where W $[\text{sec}^{-1}]$ is the width of the specified band and $N_0 W$ [W] is the average noise power in the band. Here C is defined as the maximum value of the average mutual information per second, evaluated over all values of time, and all probability distributions $p(\underline{u})$ satisfying eq. (1). The value of C increases monotonically with increasing W; note that $C \rightarrow S/(N_0 \ln 2)$ as $W \rightarrow \infty$. The proof of this theorem can be made through application of the maximum entropy theorem (Shannon & Weaver, 1949; Fano, 1961: 159).

First we note that the logarithmic function in the WSF of eq. (2) can be written in terms of Gaussian hypergeometric function F (a, b, c; z)

$$\ln (1-z) = -z \, F (1, 1, 2; z), \tag{3}$$

where $z \equiv -S/(N_0 W)$ and $\ln \equiv \log_e$. Note that the hypergeometric function can be defined through a single definite integral termed Euler's integral representation (Abramowitz & Stegun, 1965)

$$F (a, b, c; z) = \frac{\Gamma (c)}{\Gamma (b) \, \Gamma (c\text{-}b)} \int_0^1 u^{b-1}$$
$$\times (1-u)^{c-b-1} (1-uz)^{-a} \, du. \tag{4}$$

From eqs. (3) and (4) we obtain the identity

$$\ln (1-z) = -z \int_0^1 (1-uz)^{-1} \, du, \tag{5}$$

where $\Gamma (\zeta)$ is the complete gamma function with argument ζ. Hence, from eqs. (2) and (5) with $z = -S/(N_0 W)$ and $\tau \equiv (N_0/S)(1/u - 1)$, we derive an integral formula

$$C = K \int_0^\infty (\tau + N_0/S)^{-1}$$
$$\times (\tau + N_0/S + W^{-1})^{-1} \, d\tau, \tag{6}$$

where $K = (\ln 2)^{-1}$. Note that the dimension of τ is second. Mathematically eq. (6) is a correlation between $(\tau + N_0/S)^{-1}$ and $(\tau + N_0/S + W^{-1})^{-1}$. Thus one can readily understand from eq. (6) that the channel capacity per second increases with increasing either W or S/N_0. Since the capacity C [bit/sec] indicates the speed of information transmission, the integrand of eq. (6) may become an analog of the instantaneous (time-dependent) acceleration $\alpha (\tau)$ [bit/sec^2] of information transmission in band-limited, additive white Gaussian channels:

$$\alpha(\tau) \equiv K(\tau + N_0/S)^{-1}(\tau + N_0/S + W^{-1})^{-1}. \tag{7}$$

The acceleration given by eq. (7) decreases monotonically with increasing τ; as $\tau \rightarrow \infty$ it exhibits square-law decay: $\alpha (\tau) \sim K \, \tau^{-2}$.

In analogy to the Newton's second law of motion, eq. (7) might be identifiable with an instantaneous "force" per unit mass, though one could not define explicitly the concept of mass in the information-theoretic context. The definition relation of eq. (7) permits of an instantaneous velocity of information transmission:

$$v (\tau) = \int_0^\tau \alpha (\tau) \, d\tau. \tag{8}$$

Substitution of eq. (7) into eq. (8) yields

$$v\ (\tau) = W\ \log_2\ \{[1+S/(N_0W)]$$
$$\times (\tau + N_0/S)(\tau + N_0/S + W^{-1})^{-1}\}. (9)$$

Obviously $C = v\ (\infty) > v\ (\tau)$, which indicates that as $\tau \to \infty$ the information velocity approaches monotonically the channel capacity per second.

According to Shannon's theorem of channel coding (Shannon & Weaver, 1949) the information transmission rate (the information rate for short), R[bit/sec], must not exceed C, i. e., $R < C$, provided that the bit error rate can be suppressed in an optional fashion. Since from eq. (9) v (τ) shows the univalent correspondence to τ, with $R \equiv v\ (\tau)$ we can obtain an expression of τ in terms of R:

$$\tau = 2^{R/W}\ W^{-1}/[1 + S/(N_0W)$$
$$- 2^{R/W}] - N_0/S. \qquad (10)$$

Note that as $R \to C$, $\tau \to \infty$.

Now that information velocity has been given by eq. (9), the conservation law of "mechanical energy" allows one to define an analog of the potential energy per unit mass for the information transmission:

$$U\ (\tau) \equiv \{C^2 - [v\ (\tau)]^2\}/2. \qquad (11)$$

This relation indicates that the potential energy decreases monotonically with increasing τ; eventually $U\ (\tau) \to 0$ as $\tau \to \infty$. Evidently the physical meaning of eq. (11) is consistent with what the Shannon's theorem of channel coding (Shannon & Weaver, 1949) implies.

In conclusion, through derivation of an integral representation of the WSF, it has been shown that an effective acceleration of information transmission can be introduced in the information-theoretic context. The acceleration thus defined has been shown to exhibit a long-time tail with an inverse square-law profile.

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